STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFD. TOEGEPASTE WISKUNDE

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A note on the problem of the sloping beach

by

H.A. Lauwerier

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§ 1. Introduction

The problem of the nature of the progressing waves over a uniformly sloping beach is amply discussed in the well-known book of Stoker on water waves [Stoker, 1957]. The problem has been considered first by Miche [1944] who treated the case of two-dimensional waves over beaches sloping at the angles $\pi/2n$ with n an integer. For slope angles which are rational multiples of a right angle of the special form $p\pi/2n$ with p any odd integer smaller than 2n the problem has been treated independently by Lewy [1946]. The problem has been treated since by a number of writers culminating in the work of Peters [1952] and Roseau [1952] who solved the general case of threedimensional waves over beaches sloping at an arbitrary angle. For fuller technical and bibliographical information the reader is referred to the appropriate chapter in Stoker's book.

In this paper the general problem is solved anew by making use of a method devised by Van Dantzig [1958] in dealing with a similar type of problem. In this way the solutions are obtained in a new form which makes them perhaps more amenable to a further treatment.

The method rests in principle on the possibility of representing the solution as a Fourier integral (4.1). The boundary conditions induce a functional equation (4.7) which can be solved explicitly. There are two solutions leading to two types of progressing waves which are out of phase at infinity.

The solution of the general problem is preceded by a discussion of the special case of the reflection of three-dimensional waves against a vertical cliff. In this case the solution can be found in a very simple way.

§ 2. The problem

Let the beach be represented in cylindrical coordinates (r, φ, z) by $r > 0$, $0 < \varphi < \theta < \pi$, $-\infty < z < \infty$ where $\varphi = 0$ at the bottom and $\varphi = \theta$ at the undisturbed surface of the sea. The line $r=0$, $-\infty < z < \infty$ represents the shore. Sometimes also Cartesian coordinates will be used with $x=r\cos\varphi$ and $y=r\sin\varphi$. Our notation is the same as that of Stoker with some non-essential modifications.

Then we seek a velocity potential ϕ satisfying

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi = 0, \qquad 2.1$$

for which
$$\frac{\partial\phi}{\partial\varphi} = 0 \quad \text{at the bottom } \varphi = 0, \qquad 2.2$$

and
$$\frac{1}{r}\frac{\partial\phi}{\partial\varphi}+\frac{1}{g}\frac{\partial^2\phi}{\partial t^2} = 0 \text{ at the surface } \varphi=\Theta, \qquad 2.3$$

where g is the constant of gravity.

The velocity potential ϕ is sought in the form

$$\phi = e^{i(\sigma t+kz)}\, f(r,\varphi), \qquad 2.4$$

where σ and k are real constants with $k>0$, so that $f(r,\varphi)$ satisfies the Helmholtz equation

$$(\Delta_{r,\varphi} - k^2)\, f = 0, \qquad 2.5$$

and the boundary conditions

$$\frac{\partial f}{\partial\varphi} = 0 \quad \text{for } \varphi = 0, \qquad 2.6$$

$$\frac{1}{r}\frac{\partial f}{\partial\varphi} - mf = 0 \quad \text{for } \varphi = \theta, \qquad 2.7$$

where $m=\sigma^2/g$.

Further at infinity ϕ should behave like a progressive wave. A progressive wave ϕ_o satisfying 2.1 and 2.3 can be represented by

$$\phi_o = e^{i(\sigma t+kz)} \exp ikr\, \mathrm{sh}\{\alpha + i(\Theta-\varphi)\}, \qquad 2.8$$

where $\mathrm{ch}\,\alpha = m/k$.

We shall assume that $m>k$ so that the existence of a proper progressive wave is secured. We note that then α is real, we shall suppose $\alpha>0$, so that at the surface $\varphi=\Theta$ the velocity potential is oscillatory at infinity. The crests of the wave are at the angle β to the shore line which is determined by

$\operatorname{tg}\beta = k/\sqrt{m^2-k^2}$.

Without loss of generality we may take k=1. This is equivalent to taking k^{-1} as the unit of length. The problem can now be reformulated as follows.

To find a solution of the Helmholtz equation

$$(\Delta_{r,\varphi} -1)f=0 \qquad 2.9$$

satisfying

$$\frac{\partial f}{\partial \varphi} = 0 \qquad \text{for } \varphi = 0, \qquad 2.10$$

and

$$\frac{1}{r}\frac{\partial f}{\partial \varphi} - f\operatorname{ch}\alpha = 0 \text{ for } \varphi = \theta, \qquad 2.11$$

and for which at the surface

$$f = \exp irs\operatorname{h}\alpha + O(1) \text{ for } r \to \infty . \qquad 2.12$$

There are two solutions satisfying 2.9, 2.10 and 2.11 which are "out of phase" at infinity. A suitable linear combination of them will lead to a solution having the form of an arbitrary progressive wave at infinity. These two solutions are of the J_0 and the Y_0 type at the surface, i.e. one of them has a logarithmic singularity at r=0 whereas the other is regular at r=0.

§ 3. A vertical cliff

If $\theta = \pi/2$ the problem becomes physically that of the reflection of progressing waves against a vertical cliff. In Cartesian coordinates the problem can be formulated as follows.

To find a function f(x,y) satisfying for $x>0$, $y>0$ the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 1\right) f=0 \qquad 3.1$$

with the boundary conditions

$$\frac{\partial f}{\partial y} = 0 \qquad \text{for } y = 0, \qquad 3.2$$

and

$$\frac{\partial f}{\partial x} + f\operatorname{ch}\alpha = 0 \qquad \text{for } x = 0. \qquad 3.3$$

It is clear that

$$\int_{-\infty}^{\infty} e^{-x\,\mathrm{ch}\,w + iy\,\mathrm{sh}\,w}\,\psi(w)\,dw, \qquad 3.4$$

where $\psi(w)$ is an arbitrary function may represent a solution of 3.1. The boundary conditions 3.2 and 3.3 require that

$$\int_{-\infty}^{\infty} e^{-x\,\mathrm{ch}\,w}\,\mathrm{sh}\,w\,\psi(w)\,dw = 0,$$

and

$$\int_{-\infty}^{\infty} e^{iy\,\mathrm{sh}\,w}(\mathrm{ch}w-\mathrm{ch}\alpha)\psi(w)\,dw = 0.$$

If α is complex and if either $0 < \mathrm{Im}\,\alpha < \pi$ or $-\pi < \mathrm{Im}\,\alpha < 0$ these conditions are solved by

$$\psi(w) = \frac{\mathrm{ch}\,w}{\mathrm{ch}w-\mathrm{ch}\alpha}.$$

Hence we obtain the solution

$$f(x,y) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-x\,\mathrm{ch}w+iy\,\mathrm{sh}w}\,\frac{\mathrm{ch}w}{\mathrm{ch}w-\mathrm{ch}\alpha}\,dw. \qquad 3.5$$

If $\mathrm{Im}\,\alpha \to 0$ we obtain from 3.5 by taking either $\mathrm{Im}\alpha > 0$ or $\mathrm{Im}\alpha < 0$

$$f(x,y) = \frac{1}{2\pi i}\,{\int\!\!\!\!\!\!-}_{-\infty}^{\infty} e^{-x\,\mathrm{ch}w+iy\,\mathrm{sh}w}\,\frac{\mathrm{ch}w}{\mathrm{ch}w-\mathrm{ch}\alpha}\,dw \pm \mathrm{cth}\,\alpha\, e^{-x\,\mathrm{ch}\alpha}\cos(y\,\mathrm{sh}\alpha). \qquad 3.6$$

where the integral is a Cauchy integral with respect to $w=\pm\alpha$. In this way <u>two</u> independent solutions of the problem are obtained. By taking sum and difference we get the standard solutions

$$f_1(x,y) = e^{-x\,\mathrm{ch}\alpha}\cos(y\,\mathrm{sh}\alpha), \qquad 3.7$$

and

$$f_2(x,y) = \tfrac{1}{2}\,{\int\!\!\!\!\!\!-}_{-\infty}^{\infty} e^{-x\,\mathrm{ch}\,w+iy\,\mathrm{sh}\,w}\,\frac{\mathrm{ch}w}{\mathrm{ch}w-\mathrm{ch}\alpha}\,dw. \qquad 3.8$$

In view of

$$\tfrac{1}{2}\int_{-\infty}^{\infty} e^{-x\,\mathrm{ch}\,w+iy\,\mathrm{sh}w}\,dw = K_0(r) \qquad 3.9$$

we may deduce from 3.8 that

$$f_2(x,y) = K_0(r) + O(1) \quad \text{for } r\to 0, \qquad 3.10$$

which shows that the second solution has a logarithmic singularity at $r=0$.
The behaviour of $f_2(x,y)$ at infinity follows from 3.8 if the right-hand side is replaced by

$$\frac{1}{2}\int_{-\infty+ic}^{\infty+ic} e^{-x\,chw+iy\,shw}\,\frac{chw}{chw-ch\alpha}\,dw+\frac{1}{2}\left\{Res(w=-\alpha)+Res(w=\alpha)\right\}$$

where c is real and $0<c<\pi$. If $x=0$ and $y\to\infty$ the integral vanishes exponentially whereas the half sum of the residues gives an oscillatory contribution. Explicitly

$$f_2(0,y)=-\pi\,cth\,\alpha\,\sin(y\,sh\alpha)+O(1)\text{ for }y\to\infty. \qquad 3.11$$

From 3.8 an expression will now be derived which is given in Stoker's book. We note that

$$\frac{\partial}{\partial x}\left\{e^{xch\alpha}f_2(x,y)\right\}=e^{xch\alpha}\frac{\partial}{\partial x}K_o(r),$$

so that by integration (cf. Stoker l.c. formula 5.3.13)

$$f_2(x,y)=e^{-xch\alpha}\int_{-\infty}^{x}e^{uch\alpha}\,dK_o(\sqrt{u^2+y^2}). \qquad 3.12$$

§ 4. The general case

We shall now consider the general case 2.9, 2.10 and 2.11. According to Van Dantzig (l.c. theorem 1) the Helmholtz equation 2.9 has the general solution

$$f(r,\varphi)=\int_{-\infty}^{\infty}e^{-irshw}\left\{F_1(w+i\varphi)+F_2(-w+i\varphi)\right\}dw, \qquad 4.1$$

where F_1 and F_2 are holomorphic functions of their arguments in the strip determined by $0<\varphi<\Theta$.

The boundary condition 2.10 gives

$$\int_{-\infty}^{\infty}e^{-irshw}\,chw\left\{F_1(w)-F_2(-w)\right\}dw=0. \qquad 4.2$$

The boundary condition 2.11 gives

$$\int_{-\infty}^{\infty}e^{-irshw}\left\{(chw+ch\alpha)F_1(w+i\theta)-(chw-ch\alpha)F_2(-w+i\theta)\right\}dw=0. \qquad 4.3$$

Sufficient conditions for 4.2 and 4.3 are

$$F_2(-w)=F_1(w), \qquad 4.4$$

and

$$(chw-ch\alpha)F_2(-w+i\theta)=(chw+ch\alpha)F_1(w+i\theta). \qquad 4.5$$

If we put

$$F_1(w)=H(w),\quad F_2(w)=H(-w) \qquad 4.6$$

then $H(w)$ satisfies the functional equation

$$\frac{H(w+i\theta)}{H(w-i\theta)} = \frac{\mathrm{ch}w-\mathrm{ch}\alpha}{\mathrm{ch}w+\mathrm{ch}\alpha} . \tag{4.7}$$

It will be shown in the following section that for real α there are two independent solutions H_1 and H_2 which are free from zeros and poles in the strip $-\theta < \mathrm{Im}\, w < \theta$ and which are bounded at infinity. In particular

$$\begin{aligned} &\lim H_1(w) = \mathrm{sgn}(\mathrm{Re}\, w) &&\text{for } |\mathrm{Re}\, w| \to \infty \\ &\lim H_2(w) = 1 &&\text{for } |\mathrm{Re}\, w| \to \infty \end{aligned} \tag{4.8}$$

We have the following explicit result

$$H_1(w) = \frac{2^{\frac{1}{2}}\mathrm{sh}\tfrac{1}{2}\nu(w-\tfrac{1}{2}\pi i)}{(\mathrm{ch}\nu w+\mathrm{ch}\nu\alpha)^{\frac{1}{2}}} H_o(w) \tag{4.9}$$

$$H_2(w) = \frac{2^{\frac{1}{2}}\mathrm{ch}\tfrac{1}{2}\nu(w-\tfrac{1}{2}\pi i)}{(\mathrm{ch}\nu w+\mathrm{ch}\nu\alpha)^{\frac{1}{2}}} H_o(w) , \tag{4.10}$$

with $\nu = \pi/\theta$, and

$$H_o(w) = \exp i\int_0^\infty \frac{\sin wt}{t}\, \frac{\cos\alpha t\ \mathrm{th}\tfrac{1}{2}\pi t}{\mathrm{sh}\,\theta t}\, dt, \tag{4.11}$$

valid for $|\mathrm{Im}\, w| < \theta$.

Both $H_1(w)$ and $H_2(w)$ are meromorphic functions with simple poles and zeros. Both functions have simple poles at $w=-i\theta \pm \alpha$ and simple zeros at $w=i\theta \pm \alpha$.

For $f(r,\varphi)$ we may write in view of 4.6

$$f_{1,2}(r,\varphi) = \int_{-\infty}^{\infty} e^{-ir\mathrm{sh}w}\left\{H_{1,2}(w+i\varphi)+H_{1,2}(w-i\varphi)\right\} dw . \tag{4.12}$$

From 4.8 it follows that $f_1(r,\varphi)$ is continuous at $r=0$ and that $f_2(r,\varphi)$ has a logarithmic singularity at $r=0$. The behaviour at infinity of $f(r,\varphi)$ is determined by the residues of the poles of $H_{1,2}(w)$ at $w=-i\theta \pm \alpha$. We obtain at the surface

$$f_{1,2}(r,\varphi) \sim A_{1,2}\exp ir\mathrm{sh}\,\alpha + B_{1,2}\exp -ir\mathrm{sh}\,\alpha \tag{4.13}$$

with two different independent linear combinations.

In the special case $\theta = \frac{1}{2}\pi$ we obtain

$$H_1(w) = \frac{\text{ch}w}{\text{sh}w+i\text{ch}\alpha}, \quad H_2(w) = \frac{\text{sh}w}{\text{sh}w+i\text{ch}\alpha}. \qquad 4.14$$

By substitution of these results in 4.12 the formulae 3.7 and 3.8 of the previous section can easily be derived.

§5. Solution of the functional equation

We shall assume that $H'(w)/H(w)$ can be represented in the following way as a Fourier transform

$$H'(w)/H(w) = \int_{-\infty}^{\infty} e^{-iwt}\psi(t)dt. \qquad 5.1$$

Logarithmic differentiation of the functional equation 4.7 gives

$$\frac{H'}{H}(w+i\theta) - \frac{H'}{H}(w-i\theta) = \frac{d}{dw}\ln\frac{\text{ch}w-\text{ch}\alpha}{\text{ch}w+\text{ch}\alpha} = \frac{2\text{ch}\alpha\,\text{sh}w}{\text{ch}^2w-\text{ch}^2\alpha}. \qquad 5.2$$

Substitution of 5.1 gives

$$\int_{-\infty}^{\infty} e^{-iwt}\,\text{sh}\,\theta t\,\psi(t)dt = \frac{\text{ch}\alpha\,\text{sh}w}{\text{ch}^2w-\text{ch}^2\alpha}. \qquad 5.3$$

In order to avoid difficulties at $w = \pm\alpha$ we shall assume that α is complex. Until further notice we shall consider the case $-\pi < \text{Im}\,\alpha < 0$. Then inversion of 5.3 gives

$$\text{sh}\,\theta t\ \psi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iwt}\frac{\text{ch}\alpha\,\text{sh}w}{\text{ch}^2w-\text{ch}^2\alpha}dw. \qquad 5.4$$

From the calculus of residues it easily follows that

$$\text{sh}\,\theta t\ \psi(t) = \frac{\sin(\alpha+\frac{1}{2}\pi i)t}{2\,\text{ch}\frac{1}{2}\pi t}. \qquad 5.5$$

We note that $\psi(t)$ is an even function of t so that 5.1 becomes a cosine transform. Substitution of 5.5 gives

$$H'(w)/H(w) = \int_0^{\infty}\cos wt\,\frac{\sin(\alpha+\frac{1}{2}\pi i)t}{\text{sh}\theta t\,\text{ch}\frac{1}{2}\pi t}dt \qquad 5.6$$

If we define $H(0)=1$ integration of 5.6 gives

$$H(w) = \exp\int_0^{\infty}\frac{\sin wt}{t}\,\frac{\sin(\alpha+\frac{1}{2}\pi i)t}{\text{sh}\theta t\,\text{ch}\frac{1}{2}\pi t}dt. \qquad 5.7$$

This expression converges if $|\mathrm{Im}\, w| < \theta - \mathrm{Im}\, \alpha$.

If the expression 5.4 is substituted straight away into 5.1 we obtain

$$H'(w)/H(w) = -\frac{1}{\pi i}\int_0^\infty \frac{d}{du}\ln\frac{\mathrm{ch}u-\mathrm{ch}\alpha}{\mathrm{ch}u+\mathrm{ch}\alpha}\,du\int_0^\infty \cos wt\,\frac{\sin ut}{\mathrm{sh}\,\theta t}\,dt.$$

Since *)

$$\int_0^\infty \cos wt\,\frac{\sin ut}{\mathrm{sh}\,\theta t}\,dt = \frac{\nu}{2}\,\frac{\mathrm{sh}\,\nu u}{\mathrm{ch}\,\nu w+\mathrm{ch}\,\nu u} \tag{5.8}$$

where $\nu = \pi/\theta$ and $|\mathrm{Im}\, w| < \theta$,
this expression reduces to

$$H'(w)/H(w) = -\frac{1}{2\theta i}\int_0^\infty \frac{\mathrm{sh}\,\nu u}{\mathrm{ch}\,\nu w+\mathrm{ch}\,\nu u}\,d\ln\frac{\mathrm{ch}u-\mathrm{ch}\alpha}{\mathrm{ch}u+\mathrm{ch}\alpha} =$$

$$= \frac{1}{2\theta i}\int_0^\infty \ln\frac{\mathrm{ch}u-\mathrm{ch}\alpha}{\mathrm{ch}u+\mathrm{ch}\alpha}\,\frac{d}{dw}\left\{\frac{\mathrm{sh}\,\nu w}{\mathrm{ch}\,\nu u+\mathrm{ch}\,\nu w}\right\}du$$

so that

$$H(w) = \exp\frac{1}{2\theta i}\int_0^\infty \ln\frac{\mathrm{ch}t-\mathrm{ch}\alpha}{\mathrm{ch}t+\mathrm{ch}\alpha}\,\frac{\mathrm{sh}\,\nu w}{\mathrm{ch}\,\nu t+\mathrm{ch}\,\nu w}\,dt\ , \tag{5.9}$$

convergent for $|\mathrm{Im}|w < \theta$.

From 5.7 it follows that for Re $w \to \pm\infty$ the asymptotic behaviour of $H(w)$ is

$$\lim H(w) = \exp \pm \frac{\nu}{2}(\alpha + \tfrac{1}{2}\pi i)\ .$$

From 5.9 we may derive a result which says a little more

$$H(w) = \exp \pm \frac{\nu}{2}(\alpha+\tfrac{1}{2}\pi i) + O(\exp - |\mathrm{Re}\, w|\ ). \tag{5.10}$$

The analytic continuation of $H(w)$ can be found by expansion of $H(w)$ into an infinite product as Van Dantzig has demonstrated for his E-functions (l.c. Appendix). In fact $H(w)$ can be easily expressed in these functions.

We shall use the following Laplace transform

$$\int_0^\infty e^{-pt}\,\frac{1-\cos at}{t}\,dt = \tfrac{1}{2}\ln\left(1+\frac{a^2}{p^2}\right)\ . \tag{5.11}$$

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*) Erdélyi et al. Integral transforms I, formula 1.9.53

From 5.7 we can derive

$$H(w) = \exp 2 \int_0^\infty \frac{(1-\cos w_1 t)-(1-\cos w_2 t)}{t} \sum_{m,n} (-1)^n e^{-St} dt, \qquad 5.12$$

where
$$w_1 = w + \alpha + \tfrac{1}{2}\pi i,$$
$$w_1 = w - \alpha - \tfrac{1}{2}\pi i,$$
$$S = (2m+1)\theta + (2n+1)\tfrac{1}{2}\pi ,$$

and where m and n run through the non-negative integers. Application of 5.11 gives the formal expansion

$$H(w) = C \, \frac{\prod_1 (1+\frac{w_1^2}{S^2}) \prod_2 (1+\frac{w_2^2}{S^2})}{\prod_2 (1+\frac{w_1^2}{S^2}) \prod_1 (1+\frac{w_2^2}{S^2})} , \qquad 5.13$$

where in $\prod_1$ n is even and in $\prod_2$ n is odd. This expansion can be made convergent in the usual way by introduction of suitable exponential factors. The factor C is a constant which can be determined by the condition $H(0)=1$. It follows that $H(w)$ is meromorphic with simple poles and zeros. The poles are

$$w = -\alpha + i(-\tfrac{1}{2}\pi \pm S) \quad \text{for odd } n$$
$$w = \alpha + i(\tfrac{1}{2}\pi \pm S) \quad \text{for even } n.$$

The "nearest" pole and zero are a pole at $w=-i\theta+\alpha$ and a zero at $w=i\theta-\alpha$.

In the preceding discussion we took $-\pi < \mathrm{Im}\alpha < 0$. If, however, we consider the case $0 < \mathrm{Im}\alpha < \pi$ the only difference is that in formula 5.7 α must be replaced by $-\alpha$. The same remark applies to 5.10 and 5.13.

We shall now take α real. If

$$H^+(w) = \lim H(w) \quad \text{for } \mathrm{Im}w \to -0,$$

and $$H^-(w) = \lim H(w) \quad \text{for } \mathrm{Im}w \to +0$$

then both $H^+(w)$ and $H^-(w)$ are solutions of the original functional equation. We have from 5.7

$$H^{\pm}(w) = \exp \int_0^\infty \frac{\sin wt}{t} \, \frac{\sin(\pm\alpha + \frac{1}{2}\pi i)t}{\mathrm{sh}\,\theta t \; \mathrm{ch}\frac{1}{2}\pi t} \, dt, \qquad 5.14$$

valid for $|\mathrm{Im}\, w| < \theta$.
It follows from 5.14 that $H^{\pm}(w)$ can be written in the form

$$H^{\pm}(w) = \{G(w)\}^{\pm 1} H_o(w). \qquad 5.15$$

It is clear that

$$H_o(w) = \exp i \int_0^\infty \frac{\sin wt}{t} \frac{\cos \alpha t \;\mathrm{th}\tfrac{1}{2}\pi t}{\mathrm{sh}\,\theta t} dt. \qquad 5.16$$

For $G(w)$ we find

$$G(w) = \exp \int_0^\infty \frac{\sin wt}{t} \frac{\sin \alpha t}{\mathrm{sh}\,\theta t} dt.$$

By integration of 5.8 it easily follows that

$$G(w) = \left\{ \frac{e^{\nu(w+\alpha)}+1}{e^{\nu w} + e^{\nu\alpha}} \right\}^{\frac{1}{2}} . \qquad 5.17$$

We might also start from 5.9. If we consider $H^{+}(w)$ the argument of $\mathrm{ch}u - \mathrm{ch}\,\alpha$ is to be taken as π. Then we find

$$H^{\pm}(w) = \exp \pm \frac{\nu}{2} \int_0^\alpha \frac{\mathrm{sh}\,\nu w}{\mathrm{ch}\nu t + \mathrm{ch}\nu w} dt \cdot \exp \frac{1}{2\theta i} \int_0^\infty \ln \left| \frac{\mathrm{ch}t - \mathrm{ch}\alpha}{\mathrm{ch}t + \mathrm{ch}\alpha} \right| \cdot \frac{\mathrm{sh}\,\nu w}{\mathrm{ch}\nu t + \mathrm{ch}\nu w} dt.$$

This result gives the same expression 5.17 for $G(w)$, but for $H_o(w)$ a different expression is obtained.

$$H_o(w) = \exp \frac{1}{2\theta i} \int_0^\infty \ln \left| \frac{\mathrm{ch}t - \mathrm{ch}\alpha}{\mathrm{ch}t + \mathrm{ch}\alpha} \right| \frac{\mathrm{sh}\,\nu w}{\mathrm{ch}\nu t + \mathrm{ch}\nu w} dt . \qquad 5.18$$

The functions $H^{+}(w)$ and $H^{-}(w)$ are independent solutions of the functional equation 4.7. They may be replaced by any other linearly independent pair. We shall take linear combinations $H_1(w)$ and $H_2(w)$ which have the property that

$$\begin{aligned} H_1(w) &= \mathrm{sgn}(\mathrm{Re}\, w) + O(\exp - |\mathrm{Re}\, w|) \\ H_2(w) &= 1 + O(\exp - |\mathrm{Re}\, w|) \end{aligned} \qquad 5.19$$

as $\mathrm{Re}\, w \to \pm\infty$.
A simple calculation shows that

$$\begin{aligned} (\mathrm{ch}\nu w + \mathrm{ch}\nu\alpha)^{\frac{1}{2}} H_1(w) &= 2^{\frac{1}{2}} \mathrm{sh}\tfrac{1}{2}\nu(w - \tfrac{1}{2}\pi i)\, H_o(w) \\ (\mathrm{ch}\nu w + \mathrm{ch}\nu\alpha)^{\frac{1}{2}} H_2(w) &= 2^{\frac{1}{2}} \mathrm{ch}\tfrac{1}{2}\nu(w - \tfrac{1}{2}\pi i)\, H_o(w) \end{aligned} \qquad 5.20$$

and that

$$\begin{cases} \operatorname{sh}\nu\alpha\, H_1(w) = \operatorname{ch}\tfrac{1}{2}\nu(\alpha-\tfrac{1}{2}\pi i)\, H^+(w) - \operatorname{ch}\tfrac{1}{2}\nu(\alpha+\tfrac{1}{2}\pi i)\, H^-(w), \\ \operatorname{sh}\nu\alpha\, H_2(w) = \operatorname{sh}\tfrac{1}{2}\nu(\alpha-\tfrac{1}{2}\pi i)\, H^+(w) + \operatorname{sh}\tfrac{1}{2}\nu(\alpha+\tfrac{1}{2}\pi i)\, H^-(w). \end{cases} \qquad 5.21$$

§ 6. The surface waves

At the surface of the sea we have

$$f_{1,2}(r,\theta) = \int_{-\infty}^{\infty} e^{-irshw}\left\{H_{1,2}(w+i\theta) + H_{1,2}(w-i\theta)\right\} dw. \qquad 6.1$$

It must be noted that the integrand has poles at $w=-\alpha$ and at $w=\alpha$ so that the integral has to be interpreted as a Cauchy integral with respect to the points $w=\pm\alpha$.

If w is real, we shall write w=u, we have by virtue of 4.7

$$H(u+i\theta) + H(u-i\theta) = \frac{2\operatorname{ch}u}{\operatorname{ch}u-\operatorname{ch}\alpha}\, H(u+i\theta). \qquad 6.2$$

From 5.20 it follows that

$$\begin{aligned} H_1(u+i\theta) &= 2^{\frac{1}{2}} i(\operatorname{ch}\nu\alpha - \operatorname{ch}\nu u)^{-\frac{1}{2}} \operatorname{ch}\tfrac{1}{2}\nu(u-\tfrac{1}{2}\pi i)\, H_0(u+i\theta) \qquad 6.3 \\ H_2(u+i\theta) &= 2^{\frac{1}{2}} i(\operatorname{ch}\nu\alpha - \operatorname{ch}\nu u)^{-\frac{1}{2}} \operatorname{sh}\tfrac{1}{2}\nu(u-\tfrac{1}{2}\pi i)\, H_0(u+i\theta). \end{aligned}$$

Further 5.16 gives

$$\ln H_0(u+i\theta) = i\int_0^{\infty} \frac{\sin(u+\frac{1}{2}\pi i)t}{t}\,\frac{\cos\alpha t}{\operatorname{ch}\frac{1}{2}\pi t}\,dt + i\int_0^{\infty} \frac{\sin ut}{t}\,\frac{\cos\alpha t\,\operatorname{sh}(\frac{1}{2}\pi-\theta)t}{\operatorname{sh}\theta t\,\operatorname{ch}\frac{1}{2}\pi t}$$

The first integral on the right-hand side is a well known sine transform *). We find eventually

$$H_0(u+i\theta) = \left(\frac{\operatorname{ch}\alpha-\operatorname{ch}u}{\operatorname{ch}\alpha+\operatorname{ch}u}\right)^{\frac{1}{2}} \exp i\int_0^{\infty} \frac{\sin ut}{t}\,\frac{\cos\alpha t\ \operatorname{sh}(\frac{1}{2}\pi-\theta)t}{\operatorname{sh}\theta t\ \ \operatorname{ch}\frac{1}{2}\pi t}\,dt. \qquad 6.4$$

If this is substituted in 6.3 we find

$$H_{1,2}(u+i\theta) = \begin{matrix}\operatorname{ch}\\ \operatorname{sh}\end{matrix}\ \tfrac{1}{2}\nu(u-\tfrac{1}{2}\pi i)\left\{\frac{2(\operatorname{ch}u-\operatorname{ch}\alpha)}{(\operatorname{ch}\nu u-\operatorname{ch}\nu\alpha)(\operatorname{ch}u+\operatorname{ch}\alpha)}\right\}^{\frac{1}{2}} i\Phi(u), \qquad 6.5$$

where

$$\Phi(u) = \exp i\int_0^{\infty} \frac{\sin ut}{t}\ \frac{\cos\alpha t\ \operatorname{sh}(\frac{1}{2}\pi-\theta)t}{\operatorname{sh}\theta t\ \ \operatorname{ch}\frac{1}{2}\pi t}\,dt. \qquad 6.6$$

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*) Cf. Erdélyi l.c. 2.9.46

The solution is now given by 6.1, 6.2, 6.5 and 6.6.
Explicitly

$$f_{1,2}(r,\theta) = 2i \int_{-\infty}^{\infty} e^{-irshu} \frac{chu}{chu-ch\alpha} \left\{ \frac{2(chu-ch\alpha)}{(ch\,\nu u-ch\,\nu\alpha)(chu+ch\alpha)} \right\}^{\frac{1}{2}} \cdot \tag{6.7}$$
$$\cdot \begin{matrix} ch \\ sh \end{matrix} \tfrac{1}{2}\nu\,(u-\tfrac{1}{2}\pi i)\,\Phi(u)du.$$

We note that in the special case of the vertical cliff where $\nu=2$ the function Φ equals unity.

Also in the special case $\theta=\pi$ which is often called the dock problem simplifications occur.
From 6.6 it follows that

$$\Phi(u) = \exp -i \int_0^{\infty} \frac{\sin ut}{t} \frac{\cos \alpha t}{1+ch\pi t} dt . \tag{6.8}$$

In view of Erdélyi l.c. formula 1.9.6 this can be transformed into

$$\Phi(u) = \exp \frac{1}{2\pi i} \int_{\alpha-u}^{\alpha+u} \frac{t}{sht} dt . \tag{6.9}$$

In this case 6.7 becomes with $\theta=\pi$

$$\left\{ \begin{aligned} f_1(r,\theta) &= 2i \int_{-\infty}^{\infty} e^{-irshu} \frac{chu}{chu-ch\alpha} \frac{ch\frac{1}{2}u-ish\frac{1}{2}u}{(chu+ch\alpha)^{\frac{1}{2}}} \Phi(u)du \\ f_2(r,\theta) &= 2i \int_{-\infty}^{\infty} e^{-irshu} \frac{chu}{chu-ch\alpha} \frac{sh\frac{1}{2}u-ich\frac{1}{2}u}{(chu+ch\alpha)^{\frac{1}{2}}} \Phi(u)du. \end{aligned} \right. \tag{6.10}$$

Lack of time has thus far prevented the numerical computation of the surface waves for some particular case.

§ 7. Peters' solution

In order to compare our result 4.12 to that of Peters (cf. Stoker l.c. 5.4.25) we shall also give his solution. Peters finds in his notation i.e. for the solution of

$$(\Delta_{r,\varphi} - k^2)\, f = 0$$

in the sector $-\theta<\varphi<0$ (!) and satisfying

$$\frac{\partial f}{\partial \varphi} = 0 \qquad \text{for } \varphi = -\theta,$$

$$\frac{1}{r}\frac{\partial f}{\partial \varphi} - f = 0 \quad \text{for } \varphi = 0$$

the following formula ($z=re^{i\varphi}$, $\bar{z}=re^{-i\varphi}$)

$$f_{1,2}(r,\varphi) = \frac{1}{2\pi i}\int_{P_1,P_2} \exp(\zeta z+ \frac{k^2}{4}\frac{\bar{z}}{\zeta}) \frac{\zeta\, h(\zeta)}{(\zeta+ir_1)(\zeta+ir_2)}\, d\zeta, \qquad 7.1$$

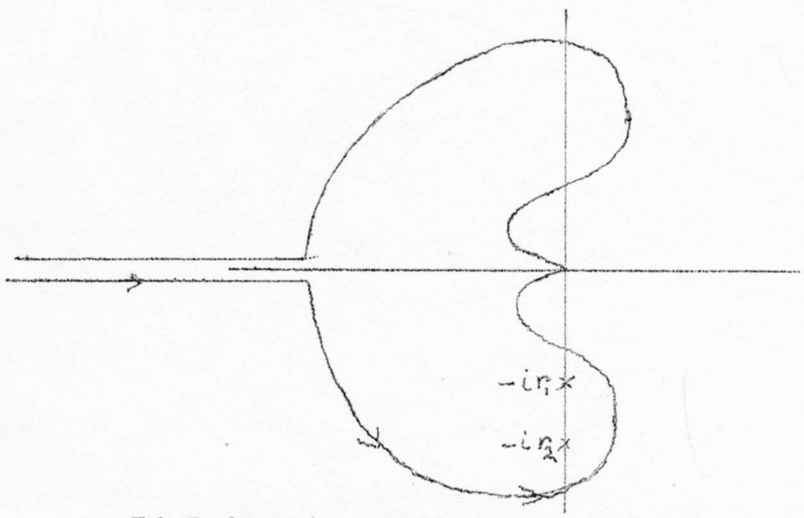
where $$h(\zeta) = \exp \frac{1}{2\pi i}\int_{-\infty}^{0} \frac{\log m(\xi^{1/\nu})}{\xi-\zeta^{\nu}}\, d\xi, \quad \arg \xi = -\pi, \qquad 7.2$$

and $$m(\zeta) = \frac{(\zeta+ir_1e^{-2i\theta})(\zeta+ir_2e^{-2i\theta})}{(\zeta-ir_1)(\zeta-ir_2)}, \qquad 7.3$$

with

$$r_1 = \tfrac{1}{2}(1+\sqrt{1-k^2}), \quad r_2 = \tfrac{1}{2}(1-\sqrt{1-k^2}), \quad \nu = \pi/\theta .$$

The contours P_1 and P_2 are indicated in the following figure. The path P_2 differs from P_1 only in the direction in which the part in the upper half-plane is traversed.



If Peters' result is translated in our notation with the dimensionless variables of § 2 we obtain by putting $k=1/\mathrm{ch}\,\alpha$, $\zeta=w/2\mathrm{ch}\,\alpha$, $\xi=(s/2\mathrm{ch}\,\alpha)^{\nu}$, $z \rightarrow z\,\mathrm{ch}\,\alpha$

$$f_{1,2}(r,\varphi) = \frac{1}{2\pi i}\int_{P_1,P_2} \exp \tfrac{1}{2}(wz+w^{-1}\bar{z}) \frac{wh(w)}{(w-ie^{\alpha})(w-ie^{-\alpha})}\, dw, \qquad 7.4$$

where

$$h(w) = \exp \frac{\nu}{2\pi i}\int_{\infty e^{-i\theta}}^{0} \frac{s^{\nu-1}}{s^{\nu}-w^{\nu}} \log \frac{(s+ie^{-2i\theta+\alpha})(s+ie^{-2i\theta-\alpha})}{(s-ie^{\alpha})(s-ie^{-\alpha})}\, ds. \qquad 7.5$$

The result 7.4 with 7.5 is still rather different from our result 4.12. The auxiliary function 7.5 is clearly related in some way to our auxiliary function H(w) if this is written in the form 5.17. A better correspondence is obtained if in 7.4 and 7.5 w and s are replaced by exponentials.

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